Consider the unsteady, 2D heat-conduction equation:

\[ \frac{\partial T}{\partial t} = \kappa \left( \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} \right) = \kappa \nabla^2 T \]

(a) Can this equation be written in characteristic/wave form? Discuss the character of this equation in terms of hyperbolic, parabolic, elliptic behavior. **NOTE:** Only carry out full Eigenvalue Method analysis if time permits.

(b) What initial conditions and boundary conditions (in x, y, and t) are required to solve this problem?

Next, consider a Finite Difference Approximation (FDA) for the two-dimensional 'Laplacian' that includes horizontal and vertical neighbors as well as the "corner points":

\[ \nabla^2 T = \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} \approx \text{FDA} \]

\[ \text{FDA} = \Delta T_{i,j} + B(T_{i+1,j} + T_{i-1,j} + T_{i,j+1} + T_{i,j-1}) + C(T_{i+1,j+1} + T_{i+1,j-1} + T_{i-1,j+1} + T_{i-1,j-1}) \]
Assume that $\Delta x = \Delta y = h$.

In order for the FDA to be consistent and 2nd-order accurate, the FDA should reduce to:

$$\text{FDA} = \nabla^2 T + K^* h^2 \nabla^2 (\nabla^2 T) + \ldots \text{ (higher order terms)}$$

where, $\nabla^2 (\nabla^2 T) = \frac{\partial^4 T}{\partial x^4} + 2\frac{\partial^4 T}{\partial x^2 \partial y^2} + \frac{\partial^4 T}{\partial y^4} = T_{xxxx} + 2T_{xxyy} + T_{yyyy}$

and thus, $\text{FDA} \to \nabla^2 T$ as $h \to 0$.

(c) Using Taylor Series expansions for the neighboring $T$ values in the FDA (including the corner points), determine the coefficients $A$, $B$, and $C$. Show that the resulting FDA for the two-dimensional Laplacian including ‘corner points’ is given by:

$$\nabla^2 T \approx \frac{1}{6} \left[ -20T_{i,j} + 4(T_{i+1,j} + T_{i-1,j} + T_{i,j+1} + T_{i,j-1}) + (T_{i+1,j+1} + T_{i+1,j-1} + T_{i-1,j+1} + T_{i-1,j-1}) \right]$$

(d) What numerical method would you suggest for obtaining the steady-state solution to this problem? Briefly outline the solution method (explicit, implicit, iterative, etc.). Include an outline and discussion of the system of equations to be solved at each timestep/iteration. Be as specific as possible.