In recent years scientists and engineers have been trying to construct artificial materials with properties not realizable in natural materials. One such family of new materials are materials which can easily undergo dilatations but are resistant to shear. This exam concerns these materials.

A smooth deformation $y : \Omega \rightarrow \mathbb{R}^3$ has gradient $F = \nabla y$, $\det F > 0$. As we know $F$ can be expressed uniquely as $F = RU$ where $R \in SO(3)$ and $U$ is positive-definite and symmetric. Also, $U$ can be further uniquely decomposed as $U = JS$ where $J = \det F$ and $\det S = 1$. Since $\det S = 1$, it preserves volume and so it can be considered a shear. Thus a scalar measure of shear is $s = |S - I|$. Here, $|A|$ is defined, as usual, by $|A| = (A : A)^{1/2} = (\tr AA^T)^{1/2}$, or, in rectangular Cartesian components $|A| = (A_{ij}A_{ij})^{1/2}$. Throughout this problem assume that $\Omega$ is connected.

The motivation behind the discovery of the new family of materials described above is the following schematic graph, where $J$ and $s$ are defined above, and the clouds denote roughly the domains associated to some standard classes of materials. The $*$ denotes the domain of the new family of materials which are resistant to shear ($s \approx 0$) but can undergo large dilatations. (Note that foams, rubber and also $*$ can undergo small shears/dilatations as well, and this is not shown in the Figure 1 to avoid overlapping clouds.)

![Figure 1: Response of some standard classes of materials (schematic) and the new family $\ast$.](image)

1. Suppose a material undergoes a large deformation $y : \Omega \rightarrow \mathbb{R}^3$ with $s = 0$ and suppose $\Omega$ contains the origin. Consider a small cube $C \subset \Omega$ centered at the origin. Draw $C$ and a typical picture of the shape of the the deformed cube $y(C)$.

We should mention that such material indicated by $*$ has been realized. It was designed by combining tiny rigid and flexible links in a unit cell in just the right way, so that a periodic array of such unit cells exhibits the response of $\ast$. In the rest of this problem we do not investigate how such a material can be made but we discuss its response.

2. Consider an ideal material of this type that satisfies exactly $s = 0$ for all deformations and at every $x \in \Omega$. Show that a smooth deformation $y : \Omega \rightarrow \mathbb{R}^3$ satisfies $s = 0$ if and only if

$$\nabla y(x)^T \nabla y(x) = \alpha(x)^2 I, \quad x \in \Omega. \quad (1)$$
for some $\alpha(x) > 0$.

In fact, under very mild conditions of smoothness the nonlinear differential equations (1) can be solved completely. We are not asking you to derive these solutions, but answer is that $y : \Omega \to \mathbb{R}^3$ satisfies (1) if and only if it is given by one of the two formulas below:

\[(A) \quad y(x) = \kappa Rx + c, \quad \tag{2}\]

\[(B) \quad y(x) = c - \frac{\kappa R(x - x_0)}{|x - x_0|^2}, \quad \tag{3}\]

where $R \in \text{SO}(3)$, $c, x_0 \in \mathbb{R}^3$, $\kappa > 0$.

3. Show that the two formulas (2) and (3) satisfy (1). Calculate $\alpha(x)$ in each case. If it is helpful you can of course use rectangular Cartesian components.

Note that (3) is not a homogeneous deformation. Surprisingly, there is a very limited class of non homogeneous deformations that are possible in this new material.

4. A deformation given by the formula (3) with $c = x_0 = 0$ and $\kappa = 4$ is shown in Figure 2. The reference configuration is on the left and the deformed configuration is on the right. A particular point $x \in \partial \Omega$ is shown in Figure 2. Put a dot (with label) at the approximate location of $y(x)$. (Be sure to pass in this sheet with your exam.)

Figure 2: A deformation given by the formula (3) with $c = x_0 = 0$ and $\kappa = 4$. The reference configuration is on the left and the deformed configuration is on the right.

A material of this type that only undergoes deformations of the type $s = 0$ can be considered a highly constrained material. As you may recall, for constrained materials, the constitutive relations
are modified by including a constraint stress. For example, for a homogeneous elastic material the constitutive relation for the Cauchy stress is written

\[ \sigma = \sigma_c + \tilde{\sigma}(F). \]  

(4)

where \( \sigma_c \) is the constraint stress. The constitutive part of the stress \( \tilde{\sigma} \) is subject to the usual restrictions, such as the principal of material frame-indifference. You are probably most familiar with the case of an incompressible elastic material in which \( \sigma_c = -pI \). Recall also that \( p \) is not determined by the deformation gradient but rather it is available as a new unknown function, that can be used to help satisfy the equations of motion, but with the additional constraint \( \det \nabla y = 1 \).

As a warm-up, let us review the derivation of the constraint stress for incompressible elastic materials. The theory behind the constraint stress is that it is the most general form of the stress for which the stress power vanishes for all motions satisfying the constraint. The stress power of the constraint stress is \( \sigma_c \cdot d \) where \( d = (1/2)(\nabla_y v + (\nabla_y v)^T) \). As indicated by the notation \( \nabla_y \) is the Eulerian gradient and \( v(y, t) \) is the Eulerian velocity field. Recall also the fundamental relations between Eulerian and Lagrangian descriptions of motion:

\[ \dot{y}(x, t) = v(y(x, t), t), \quad \dot{y}(y^{-1}(y, t), t) = v(y, t). \]  

(5)

For incompressible materials the constraint is \( \text{div}_y v = 0 \), which is \( \text{tr} d = 0 \). Thus, \( \sigma_c \) must satisfy \( \sigma_c \cdot d = 0 \) for all \( d \) satisfying \( \text{tr} d = 0 \). As usual, the dot product between tensors is \( A \cdot B = \text{tr} AB^T \).

5. Suppose \( \sigma_c = \sigma_c^T \). Show that \( \sigma_c \cdot d = 0 \) for all \( d = d^T \) satisfying \( \text{tr} d = 0 \) if and only if \( \sigma_c = -pI \) for some \( p \).

Now we turn to case of a material satisfying the constraint \( s = 0 \) (but not incompressibility, of course).

6. Consider a constrained material satisfying the constraint \( s = 0 \). Find the most general form of the constitutive part of the Cauchy stress \( \tilde{\sigma}(F) \) consistent with the principle of material frame indifference.

7. Find the form of the constraint stress \( \sigma_c \) for this material.