Problem 1: Consider a plate being pulled out from a liquid bath with a constant speed $U$ and at an angle $\theta$ with the horizontal. As the plate leaves the bath, it entrains a layer of liquid. Consider the situation where: the thickness of the liquid layer $\delta$ is constant; there is no net mass flow of liquid (as much liquid is being pulled up by the plate as is falling back by gravity); and the velocity of the liquid in contact with the plate is $V_x = U$ (parallel to the plate). Assume the flow is steady and two-dimensional, and the fluid Newtonian with constant properties. Neglect any shear force exerted by the air on the liquid surface. Assume the air pressure is negligible.

(a) Find an expression for the thickness of the liquid layer, as a function of: the plate velocity $U$, the angle $\theta$, the liquid density $\rho$ and viscosity $\mu$, and the gravitational acceleration $g$.

(b) Draw the profiles of the velocity and shear stress in the liquid, and comment on their compatibility with the boundary conditions.

(c) Write expressions for the relevant non-dimensional numbers, explaining your reasoning.

(d) Now consider the case in which the surface tension at the air-liquid interface is not negligible. How does the flow change? How does the set of relevant non-dimensional number change?
Problem 2: Consider a two-dimensional axisymmetric flow with circular streamlines and azimuthal velocity profile \( u_\theta(r) \).

(a) Ignoring the effects of viscosity, find an expression for the force per unit volume acting on a fluid particle orbiting at radius \( r \).

(b) Suppose the particle is suddenly displaced a small distance in the radial direction to \( r + \Delta r \). Suppose this displacement is inviscid in nature such that the particle experiences no torque with respect to the center of its orbit. Find the azimuthal velocity of the particle at its new position.

(c) Consider the balance of the particle’s centripetal acceleration to forces acting upon it at its new position. In terms of the velocity profile \( u_\theta(r) \), radius \( r \), and displacement \( \Delta r \), derive an expression for the net force per unit volume acting upon the particle at its new position. Consider the limit \( \Delta r \ll r \). When will the net force on the particle act to restore it to its original position?

(d) For \( \Delta r \ll r \), find a simple criterion in terms of the local angular velocity and the local vorticity governing when the net force is restoring (in which case the flow is stable to centrifugal perturbations). Apply this criterion to evaluate the centrifugal stability of both a free (potential) vortex as well as a forced vortex (solid body rotation). Briefly comment on the physical significance of your results.

(e) Consider two-dimensional boundary layer flows over slightly concave and convex surfaces. Which do you expect to be unstable? Can you sketch the flow that develops in the unstable case? (Note: although viscous effects govern the base flow, assume small perturbations behave inviscidly.)