Consider the two-dimensional, steady-state, incompressible, Parabolized Navier-Stokes (PNS) equations:

\[
\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \\
\frac{\partial u^2}{\partial x} + \frac{\partial uv}{\partial y} = \frac{\mu}{\rho} \frac{\partial^2 u}{\partial y^2} \\
\frac{\partial uv}{\partial x} + \frac{\partial v^2}{\partial y} = -\frac{1}{\rho} \frac{\partial p}{\partial y} + \frac{\mu}{\rho} \frac{\partial^2 v}{\partial y^2}
\]

Furthermore, assume that \( u > 0 \) everywhere.

1. Characterize the systems of PDEs (i.e. hyperbolic, parabolic, elliptic, or mixture).

**HINTS:** First re-write the equations in non-conservative form (use the continuity equation to simplify). Then convert the system of three PDEs, to a system of four first-order PDEs. These steps will allow you to write the system in the following form:

\[
[A] \frac{\partial Q}{\partial x} + [B] \frac{\partial Q}{\partial y} = RHS
\]

\[
Q = \begin{bmatrix} u \\ v \\ p \\ c \end{bmatrix} \quad c \equiv \frac{\partial u}{\partial y}
\]

Finally, using the continuity equation, note that:

\[
\frac{\partial c}{\partial x} = \frac{\partial}{\partial x} \left( \frac{\partial u}{\partial y} \right) = \frac{\partial}{\partial y} \left( \frac{\partial u}{\partial x} \right) = \frac{\partial}{\partial y} \left( -\frac{\partial v}{\partial y} \right) = -\frac{\partial^2 v}{\partial y^2}
\]

With these steps and relations, proceed to characterize the system.
2. Given the character of the PDEs, in which co-ordinate direction(s) must information be coupled? Is a marching solution method appropriate?

HINT: Ideally, you should use your solution to question 1 to answer this question. Otherwise, you could answer this question by studying the above PNS equations and using your CFD intuition.

3. Develop an explicit method to solve the PNS equations. Ideally, this method should be an optimal one based on your answers to questions 1 and 2. Be as specific as possible and briefly discuss the advantages and disadvantages of your method.

HINT: Start from the original PNS equations (in conservative form) and proceed to develop your method in matrix-vector form:

\[
\frac{\partial E}{\partial x} + \frac{\partial F_{\text{conv}}}{\partial y} + \frac{\partial^2 F_{\text{visc}}}{\partial y^2} = 0
\]

\[
E = \begin{bmatrix} u^2 + \omega p \\ u v \end{bmatrix}, \quad F_{\text{conv}} = \begin{bmatrix} v \\ u v \\ v^2 + p \end{bmatrix}, \quad F_{\text{visc}} = -\mu \begin{bmatrix} 0 \\ \frac{u}{4} \\ \frac{-3}{4} v \end{bmatrix}
\]

Note that a pressure-term has been artificially added to the x-momentum equation (with coefficient \(0 < \omega < 1\)). Assume \(\omega\) is a given constant.

4. Discuss the stability behavior of your explicit method, developed in question 3.

You do not need to perform stability analysis. Rather, discuss the stability behavior of the following model equation, and how this relates to your method for the PNS equations:

\[
\frac{\partial u}{\partial x} + c \frac{\partial u}{\partial y} = \mu \frac{\partial^2 u}{\partial y^2}
\]

5. Discuss the order of accuracy of your method in terms of grid spacing \(\Delta x\) and \(\Delta y\). You do not need to perform full truncation error analysis.